UNCLASSIFIED

AD 92752

Armed Services Technical Information Agency

Reproduced by
DOCUMENT SERVICE CENTER
KNOTT BUILDING, DAYTON, 2, OHIO

UNCLASSIFIED

AN EXPANSION OF THE KUMMER FUNCTION

$${}_1F_1(\alpha; \tfrac{1}{2}; \beta x^2)$$

By

Gordon Kent

March 5, 1956

Office of Naval Research

Contract N5ori-76

Task Order No. 1

NR-372-012

Technical Report

on

An Expansion of the Kummer Function $_1F_1(\alpha;\frac{1}{2};\beta x^2)$

by
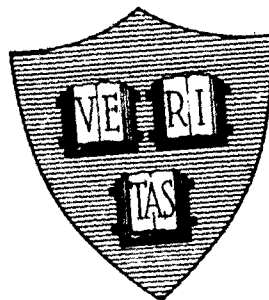
Gordon Kent

March 5, 1956

Technical Report No. 234

Cruft Laboratory

. Harvard University

Cambridge, Massachusetts

An Expansion of the Kummer Function $_1F_1(a;\frac{1}{2};\beta x^2)$

by

Gordon Kent

Division of Engineering and Applied Physics

Harvard University

Cambridge, Massachusetts

In recent years the confluent hypergeometric or Kummer functions have become useful in the treatment of a vast variety of problems in mathematical physics. The Kummer function expansion presented here was developed by the author for the solution of an electron beam problem.[1] Although its form is not the most convenient for this and for similar applications, it has the virtue that the terms of the expansion are easily calculated and are expressible in elementary functions.

The Kummer function $_1F_1(a;\frac{1}{2};\beta x^2)$ is defined by the series[2]

$$_1F_1(a;\frac{1}{2};\beta x^2) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \frac{(4\beta x^2)^n}{(2n)!} \qquad (1)$$

The behavior of the function under the conditions

$$|a| \longrightarrow \infty$$

$$|\beta| \longrightarrow 0 \qquad (2)$$

$$|a\beta| \quad \text{is finite}$$

is to be determined. Evidently, the parameter β is only a scale factor and could be eliminated; but it is more convenient to keep the parameter so that the conditions (2) do not have to be imposed on the variable, which may be

permitted to be of the order of unity without invalidating the results.

Inspection of the n'th term of the Kummer series, (1), reveals that it contains a finite sum of terms of descending orders of magnitude, specifically of orders $(4\alpha\beta)^n$, $4\beta(4\alpha\beta)^{n-1}$, $\ldots(4\beta)^{n-1}(4\alpha\beta)$. If the terms of the Kummer series, which is absolutely and uniformly convergent, can be rearranged, then these orders of magnitude can be separated. To accomplish this rearrangement, let

$$\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \sum_{k=0}^{k=n} a_k(n)\alpha^k \qquad . \tag{3}$$

and substitute (3) into (1). When the order of summation is reversed, there results the expression

$${}_1F_1(\alpha;\tfrac{1}{2};\beta x^2) = \sum_{m=0}^{\infty} b_m(x)(4\beta)^m \quad, \tag{4}$$

where

$$b_m = \sum_{n=0}^{\infty} \frac{(4\alpha\beta)^{n-m}a_{n-m}(n)x^{2n}}{(2n)!} \quad . \tag{5}$$

It can be seen by examination of (3) that

$$a_n(n) = 1$$

$$a_{n-m}(n) = 0, \text{ for } m > n \tag{6}$$

$$a_p(q) = 0, p > q$$

and the general expression for $a_{n-m}(n)$ is

$$a_{n-m}(n) = \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{k_1-1} \sum_{k_3=0}^{k_2-1} ------ \sum_{k_m=0}^{k_{m-1}-1} k_1 k_2 k_3 \ldots k_m. \tag{7}$$

With the use of the functional equation for $\Gamma(\alpha+n)$ and (6) the validity of

(7) can be proved by induction.

For computational purposes, it is more convenient to write (7) in the form

$$a_{n-m}(n) = \sum_{k=m}^{n-1} k a_{k+1-m}(k), \quad m = 1, 2, 3, \ldots \tag{8}$$

Since $a_n(n) = 1$, the expressions for $a_{n-m}(n)$ for subsequent values of m can be calculated with the aid of the following two relations from combinatorial analysis:

$$\sum_{\ell=0}^{n-1} \binom{\ell}{k} = \binom{n}{k+1} \tag{9}$$

and

$$n\binom{n}{m} = (m+1)\binom{n}{m+1} + m\binom{n}{m} \tag{10}$$

The first four of these coefficients are

$$\left. \begin{aligned} a_n(n) &= 1, \\ a_{n-1}(n) &= \binom{n}{2}, \\ a_{n-2}(n) &= 3\binom{n}{4} + 2\binom{n}{3}, \\ a_{n-3}(n) &= 15\binom{n}{6} + 20\binom{n}{5} + 6\binom{n}{4}. \end{aligned} \right\} \tag{11}$$

The first four coefficients of $(4\beta)^m$ in (4) may now be written

$$\left. \begin{aligned} b_0 &= \sum_{n=0}^{\infty} \frac{q^n}{(2n)!}, \\ \\ b_1 &= \frac{1}{4\alpha\beta} \sum_{n=0}^{\infty} \frac{q^n}{(2n)!} \binom{n}{2}, \end{aligned} \right. \tag{12}$$

$$b_2 = (\frac{1}{4\alpha\beta})^2 \sum_{n=0}^{\infty} \frac{q^n}{(2n)!}[\ 3\binom{n}{4} + 2\binom{n}{3}],$$

$$b_3 = (\frac{1}{4\alpha\beta})^3 \sum_{n=0}^{\infty} \frac{q^n}{(2n)!}[\ 15\binom{n}{6} + 20\binom{n}{5} + 6\binom{n}{4}],$$

(12)

in which the new variable $q = 4\alpha\beta x^2$. These coefficients may be expressed in closed form in terms of $b_0(q)$ and its derivatives. The expressions are

$$b_0 = \cosh\sqrt{q}\ ,$$

$$b_1 = \frac{(4\alpha\beta x^4)}{2!} \cdot b_0^{(2)}\ ,$$

$$b_2 = \frac{3(4\alpha\beta)^2 x^8}{4!} b_0^{(4)} + \frac{2(4\alpha\beta)x^6}{3!} b_0^{(3)}\ ,$$

$$b_3 = \frac{15(4\alpha\beta)^3 x^{12}}{6!} b_0^{(6)} + \frac{20(4\alpha\beta)^2 x^{10}}{5!} b_0^{(5)} + \frac{6(4\alpha\beta)x^8}{4!} b_0^{(4)}$$

(13)

where $b_0^{(k)}$ indicates the k'th derivative of $b_0(q)$ with respect to q.

It is apparent that the procedure by which (13) was developed can be applied to as many terms as desired, and that all the coefficients will be expressible as a linear combination of derivatives of $b_0(q)$.

The first four terms of the expansion of the Kummer function obtained by completing the operations indicated in (13) are

$${}_1F_1(a;\frac{1}{2};\beta x^2) = \cosh kx + (\frac{\beta x^2}{2})\left\{\cosh kx - \frac{\sinh kx}{kx}\right\}$$

$$+ \frac{1}{2!}(\frac{\beta x^2}{2})^2\left\{[\ 1 - \frac{1}{(kx)^2}]\cosh kx - [\frac{2}{3} - \frac{1}{(kx)^2}]\frac{\sinh kx}{kx}\right\}$$

$$+ \frac{1}{3!}(\frac{\beta x^2}{2})^3\left\{[\ 1 - \frac{7}{(kx)^2} - \frac{15}{(kx)^4}]\cosh kx + [\ 1 + \frac{12}{(kx)^2} + \frac{15}{(kx)^4}]\frac{\sinh kx}{kx}\right\}$$

$$+ 0(\beta^4)\ ,$$

(14)

where for convenience $k^2 = 4\alpha\beta$. Analysis shows that the terms in the $\{\ \}$

brackets vanish uniformly as $x \to 0$. For small values of x, these terms,

in the order given in (14), approach the forms

$$\left. \begin{array}{c} \dfrac{2}{3}(kx)^2 \quad , \\[2ex] \dfrac{7}{18}(kx)^2 \quad , \\[2ex] \dfrac{107}{120}(kx)^2 \quad , \end{array} \right\} \tag{15}$$

Thus it appears that in the neighborhood of the origin the approximation is

extraordinarily good.

References

1. G. Kent, "Space Charge Waves in Inhomogeneous Electron Beams," J. Appl. Phys. 25, 32-41, January, 1954.

2. W. Magnus and F. Oberhettinger, Formulas and Theorems for the Functions of Mathematical Physics, Chelsea Publishing Co., New York, 1949.

DISTRIBUTION LIST

Technical Reports

	Chief of Naval Research Department of the Navy Washington 25, C. C.
2	427
1	460
1	421
6	Director (Code 5250) Naval Research Laboratory Washington 25, D. C.
2	Commanding Officer Office of Naval Research Branch Office 150 Causeway Street Boston, Massachusetts
1	Commanding Officer Officer of Naval Research Branch Office 1000 Geary Street San Francisco 9, California
1	Commanding Officer Office of Naval Research Branch Office 1030 E. Green Street Pasadena, California
1	Commanding Officer Office of Naval Research Branch Office The John Crerar Library Building 86 East Randolph Street Chicago 1, Illinois
1	Commanding Officer Office of Naval Research Branch Office 346 Broadway New York 13, New York
3	Officer-in-Charge Office of Naval Research Navy No. 100 Fleet Post Office New York, New York
	Chief, Bureau of Ordnance Navy Department Washington 25, D. C.
1	Re4-1
1	AD-3

Technical Reports

	Commanding General Wright Air Development Center Wright–Patterson Air Force Base, Ohio
1	WCRRH
2	WCLR
1	WCLRR
2	Technical Library
2	Commander Wright Air Development Center Attn: WCREO Wright-Patterson Air Force Base, Ohio
	Commanding General Rome Air Development Center Griffiss Air Force Base Rome, New York
1	RCREC-4C
1	RCR-1
2	RCRW
	Commanding General Air Force Cambridge Research Center 230 Albany Street Cambridge 39, Massachusetts
6	CRR
1	Technical Library
2	Director Air University Library Maxwell Air Force Base, Alabama
1	Commander Patrick Air Force Base Cocoa, Florida
1	Chief, European Office Air Research and Development Command Shell Building 60 Rue Ravenstein Brussels, Belgium
1	U. S. Coast Guard (EEE) 1300 E Street, N. W. Washington, D. C.
1	Assistant Secretary of Defense (Research and Development Research and Development Board Department of Defense Washington 25, D. C.

Technical Reports

	Chief, Bureau of Aeronautics Navy Department Washington 25, D. C.
1	EL-1
1	EL-4
2	Chief, Bureau of Ships (810) Navy Department Washington 25, D. C.
1	Chief of Naval Operations Navy Department Washington 25, D. C.
1	Op-413
1	Op-20
1	Op-32
1	Director Naval Ordnance Laboratory White Oak, Maryland
2	Commander U. S. Naval Electronics Laboratory San Diego, California
1	Commander(AAEL) Naval Air Development Center Johnsville, Pennsylvania
1	Librarian U. S. Naval Post Graduate School Monterey, California
50	Transportation Officer Building 151 Squier Signal Laboratory Fort Monmouth, New Jersey Attn: Director of Research
	Commanding General Air Research and Development Command Post Office Box 1395 Baltimore 3, Maryland
3	RDTRRP
2	RDDDE
1	Commanding General(WCRR) Wright Air Development Center Wright-Patterson Air Force Base, Ohio

Technical Reports

Commanding General
Wright Air Development Center
Wright–Patterson Air Force Base, Ohio

1	WCRRH
2	WCLR
1	WCLRR
2	Technical Library

2   Commander
Wright Air Development Center
Attn: WCREO
Wright-Patterson Air Force Base, Ohio

Commanding General
Rome Air Development Center
Griffiss Air Force Base
Rome, New York

1	RCREC-4C
1	RCR-1
2	RCRW

Commanding General
Air Force Cambridge Research Center
230 Albany Street
Cambridge 39, Massachusetts

6	CRR
1	Technical Library

2   Director
Air University Library
Maxwell Air Force Base, Alabama

1   Commander
Patrick Air Force Base
Cocoa, Florida

1   Chief, European Office
Air Research and Development Command
Shell Building
60 Rue Ravenstein
Brussels, Belgium

1   U. S. Coast Guard (EEE)
1300 E Street, N. W.
Washington, D. C.

1   Assistant Secretary of Defense
(Research and Development
Research and Development Board
Department of Defense
Washington 25, D. C.

Technical Reports

5	Armed Services Technical Information Agency Document Service Center Knott Building Dayton 2, Ohio
1	Office of Technical Services Department of Commerce Washington 25, D. C.
1	Commanding Officer and Director U. S. Underwater Sound Laboratory New London, Connecticut
1	Federal Telecommunications Laboratories, Inc. Technical Library 500 Washington Avenue Nutley, New Jersey
1	Librarian Radio Corporation of America RCA Laboratories Princeton, New Jersey
1	Sperry Gyroscope Company Engineering Librarian Great Neck, L. I., New York
1	Watson Laboratories Library Red Bank, New Jersey
1	Professor E. Weber Polytechnic Institute of Brooklyn 99 Livingston Street Brooklyn 2, New York
1	University of California Department of Electrical Engineering Berkeley, California
1	Dr. E. T. Booth Hudson Laboratories 145 Palisade Street Dobbs Ferry, New York
1	Cornell University Department of Electrical Engineering Ithaca, New York
1	University of Illinois Department of Electrical Engineering Urbana, Illinois

Technical Reports

1	Johns Hopkins University Applied Physics Laboratory Silver Spring, Maryland
1	Professor A. von Hippel Massachusetts Institute of Technology Research Laboratory for Insulation Research Cambridge, Massachusetts
1	Director Lincoln Laboratory Massachusetts Institute of Technology Cambridge 39, Massachusetts
1	Mr. A. D. Bedrosian Room 22A-209 Signal Corps Liaison Office Massachusetts Institute of Technology Cambridge, Massachusetts
1	Mr. Hewitt Massachusetts Institute of Technology Document Room Research Laboratory of Electronics Cambridge, Massachusetts
1	Stanford University Electronics Research Laboratory Stanford, California
1	Professor A. W. Straiton University of Texas Department of Electrical Engineering Austin 12, Texas
1	Yale University Department of Electrical Engineering New Haven, Connecticut
1	Mr. James F. Trosch, Administrative Aide Columbia Radiation Laboratory Columbia University 538 West 120th Street New York 27, New York
1	Dr. J.V.N. Granger Stanford Research Institute Stanford, California
1	Library Central Radio Propagation Laboratory National Bureau of Standards Boulder, Colorado

Technical Reports

1	Library of the College of Engineering New York University University Heights Library University Heights 33, New York
1	Documents and Research Information Section Raytheon Manufacturing Company Engineering Equipment Division 148 California Street Newton 58, Massachusetts
1	Professor Henry G. Booker School of Electrical Engineering Cornell University Ithaca, New York
1	M. A. Krivanich, Technical Advisor to Deputy Chief Ballistics Research Laboratory White Sands Annex White Sands P.G., New Mexico
1	Doris P. Baster Head, Document Section Technical Information Division Naval Research Laboratory Washington 25, D. C.
1	Dr. C. H. Papas Department of Electrical Engineering California Institute of Technology Pasadena, California
1	Airborne Instrument Laboratory Mineola New York
1	Radiation Laboratory Johns Hopkins University 1315 St. Paul Street Baltimore 2, Maryland
1	Mr. Robert Turner General Electric Company Advanced Electronics Center Cornell University Ithaca, New York

Technical Reports

1	Acquisitions Officer ASTIA Reference Center The Library of Congress Washington 25, D. C.
1	Librarian National Bureau of Standards Library Connecticut Avenue and Van Ness Street, N.E. Washington 25, D. C.
1	Secretary Working Group on Semiconductor Devices, AGET 346 Broadway, 8th Floor New York 13, N. Y.
1	Professor R. E. Norberg Washington University St. Louis, Missouri